Explicit Lower Bounds on the Outage Probability of Integer Forcing over $N_r \times 2$ Channels

Elad Domanovitz

Joint work with Uri Erez

November 10th, 2017
ITW, Kaohsiung, Taiwan

Introduction

- Single-user Open-loop (infinite number of users) MIMO ($N_r \geq 2$) compound (worst-case) setup
- Integer-forcing as a practical scheme
- Upper and lower bounds of integer-forcing on outage probability
- Extension 1: Symmetric-rate (statistical) $N_r \times 2$ ($N_r \geq 2$) Rayleigh MAC with minimal feedback
- Extension 2: lower bound of symmetric-rate (statistical) 1×2 Rayleigh MAC with minimal feedback

Introduction

- The Single-User Multiple-Input Multiple-Output (MIMO) Gaussian channel has been the focus of extensive research

$$\boldsymbol{y}_c = \mathbf{H}_c \boldsymbol{x}_c + \boldsymbol{z}_c,$$

- $\mathbf{x}_c \in \mathbb{C}^{N_t}$ is the channel input vector
- $\mathbf{y}_c \in \mathbb{C}^{N_r}$ is the channel output vector
- $\mathbf{H}_c$ is an $N_r \times N_t$ complex channel matrix
  $\rightarrow$ Fixed over entire block length
- $\boldsymbol{z}_c \sim \mathcal{CSCN}(0, \mathbf{I})$
- Power constraint: $\mathbb{E}(\boldsymbol{x}_c^H \boldsymbol{x}_c) \leq N_t \cdot \mathrm{SNR}$

Introduction

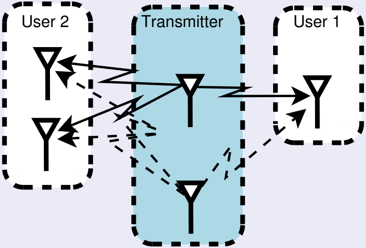
- The MIMO Gaussian broadcast channel has also been widely studied for well over a decade now:

$$\boldsymbol{y}_c^i = \mathbf{H}_c^i \boldsymbol{x}_c + \boldsymbol{z}_c^i$$

- Private (only) Messages vs. Common (only) Messages
  - Capacity is known for both scenarios ✓
  - Practical schemes?
    - Private Message ✓ (DPC: Tomlinson...)
    - Common Message?
      $\Longrightarrow$ Single user: SVD or QR+SIC
      $\Longrightarrow$ Two users: Joint triangularization (Khina et al., '12)
      $\Longrightarrow$ Moderate # of users: non-optimal extensions (Khina et al., '12)
      $\Longrightarrow$ Infinite # of users (knowing only WI-MI): Approximate joint triangularization is not very good $\Longrightarrow$ **The focus of this talk**

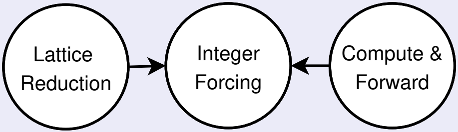
Motivation for Compound MIMO Setting

- We're interested a scheme that is:
  - Practical
    - Linear complexity in the block length
    - Uses off-the-shelf SISO codes
  - Has provable good performance guarantees
  - Universal: Is good for all channels with same WI-MI (compound channel setting), i.e., $\mathbf{H}_c \in \mathbb{H}(C_{\mathrm{WI}})$
- Universal $\Longrightarrow$ needs to deal with DoF mismatch

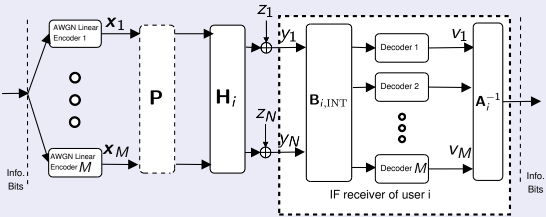


Candidate Scheme for Compound MIMO Setting: Integer Forcing

- Equalization scheme introduced by Zhan '14, et. al.



- Idea: Decode linear combination of messages $\Longrightarrow$ Invert



Candidate Scheme for Compound MIMO Setting: Integer Forcing

- What is already known?
- Ordentlich & Erez '15: using algebraic space-time precoding
  - 🙂 A linear Non-Vanishing Determinant (NVD) precoder achieves the mutual information up to a constant gap for any channel
  - 🙁 Guaranteed gap to capacity is quite large
- D. '17: using random (Haar measure) precoded (space-only precoding)
  - 🙂 Bound on the outage probability depends *only* on the gap-to-capacity and number of antennas
  - 🙁 Empirical performance still **much** better than the bound

Can we obtain a lower bound?

Lower Bound on the Outage of Precoded IF

- How does IF behaves compared to ML?
- Simple bound
  - Consider same (random) precoded but (independent) Gaussian codebooks of equal rate
  - Since codebooks are independent can be viewed as MIMO *MAC*

MIMO MAC bound [Zhan et al. '14]

- Let $\mathbf{H}_S$ denote the submatrix of $\mathbf{H}_{\text{eff}} = \mathbf{H}_c \mathbf{P}_c$ formed by taking the columns with indices in $S \subseteq \{1, 2, ..., N_t\}$
- For ML decoder, the maximal achievable symmetric rate

$$C_{\text{sym}} = \min_{S \subseteq \{1,2,\ldots,N_t\}} \frac{N_t}{|S|} \log\det\left(\mathbf{I}_{N_r} + \mathbf{H}_S \mathbf{H}_S^H\right)$$

Explicit Expressions for $N_r \times 2$

Theorem 2 (new converse)

For a randomly precoded $N_r \times 2$ compound MIMO channel with white-input mutual information C and $N_r \geq 2$, we have

$$P^{\mathrm{WC}}_{\mathrm{out,C_{sym}}}(C, \Delta C) = 1 - \sqrt{1 - 2^{-\Delta C}} \approx \frac{1}{2} 2^{-\Delta C}$$

Theorem 1 (achievable) - D. '17

For any $N_r \times 2$ complex channel $\mathbf{H}_c$ with white-input mutual information $C > 1$, i.e., $\mathbf{D} \in \mathbb{D}(C)$, and for randomly precoded $\mathbf{P}_c$ (which induces a real-valued precoding matrix $\mathbf{P}$), we have

$$P^{\mathrm{WC}}_{\mathrm{out,IF-SIC}}(C, \Delta C) \leq 81\pi^2 2^{-\Delta C},$$

for $\Delta C > 1$

Same exponent, very different constant

Lower and upper bounds for $N_r \times 2$ MIMO channel

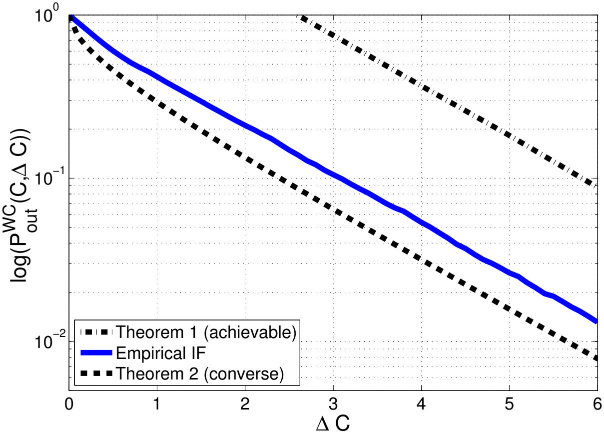


Figure: Theorem 1 and Theorem 2 for $N_r \times 2$ MIMO channels ($N_r \geq 2$) with mutual information $C = 14$.

Sketch Of Proof

- For $N_r \times 2$ the SVD of the precoded channel

$$\mathbf{H}_{\text{eff}} = \mathbf{H}_c\mathbf{P}_c = \mathbf{U}_c \begin{bmatrix} \sqrt{\rho_1} & 0 & 0 & \cdots & 0 \\ 0 & \sqrt{\rho_2} & 0 & \cdots & 0 \end{bmatrix}^H \mathbf{V}_c^H\mathbf{P}_c,$$

- $\mathbf{P}_c$ is drawn from the CUE (Haar measure) $\Longrightarrow$ $\mathbf{V}_c^H\mathbf{P}_C$ has same probability as $\mathbf{P}_c$
- Taking k columns from $\mathbf{H}_c\mathbf{P}_c$ equals to multiplying $\mathbf{H}_c$ with k columns of $\mathbf{P}_c$
- For $N_r \times 2$, $C_{\text{sym}} = \min\left(C(\{1\}), C(\{2\}), C\right)$
- $C(\{1\}) = 2\log\left(1 + \begin{bmatrix} \mathbf{P}_{1,1} \\ \mathbf{P}_{2,1} \end{bmatrix}^H \begin{bmatrix} \rho_1 & \mathbf{0} \\ \mathbf{0} & \rho_2 \end{bmatrix} \begin{bmatrix} \mathbf{P}_{1,1} \\ \mathbf{P}_{2,1} \end{bmatrix}\right)$

$$= 2\log\left(1 + \rho_1\mathbf{P}_{1,1}^H\mathbf{P}_{1,1} + \rho_2\mathbf{P}_{2,1}^H\mathbf{P}_{2,1}\right)$$

Sketch Of Proof

- $\mathbf{P}_{1,1}$ and $\mathbf{P}_{2,1}$ form a vector in a unitary matrix$\Longrightarrow$ $\mathbf{P}_{1,1}^H\mathbf{P}_{1,1} + \mathbf{P}_{2,1}^H\mathbf{P}_{2,1} = 1$
- $\Pr\left(C(\{1\}) < R|\, C\right) = \Pr\left(|\mathbf{P}_{1,1}|^2 < \frac{2^{R/2}-1-\rho_2}{\rho_1-\rho_2}\right)$
- Narula et al. '09 - squared norm of an entry in 2×2 unitary matrix drawn from the CUE is uniformly distributed over $[0, 1]$
- Recall $\rho_1 = \frac{2^C}{1+\rho_2} - 1$ we have

$$(*)\ \Pr\left(C(\{1\}) < R)|C\right) = \frac{2^{R/2} - 1 - \rho_2}{\frac{2^C}{1+\rho_2} - 1 - \rho_2}$$

- By symmetry $\Pr\left(C(\{2\}) < R)\right) = \Pr\left(C(\{1\}) < R)\right)$
- We show that the events $\{C(\{1\}) < R\}$ and $\{C(\{2\}) < R\}$ are disjoint

Sketch Of Proof

- We thus have

$$P^{\mathrm{WC}}_{\mathrm{out,C_{sym}}}(C,R) = \max_{0\le\rho_2\le 2^{C/2}-1} 2\cdot\frac{2^{R/2}-1-\rho_2}{\frac{2^C}{1+\rho_2}-1-\rho_2}.$$

- The derivative of the expression that is maximized with respect to ρ_2 is zero for (and only for)

$$\rho_2^* = 2^{-R/2-1}\left(2^{C+1}-2^{R/2+1}-2\sqrt{2^{2C}-2^{C+R}}\right),$$

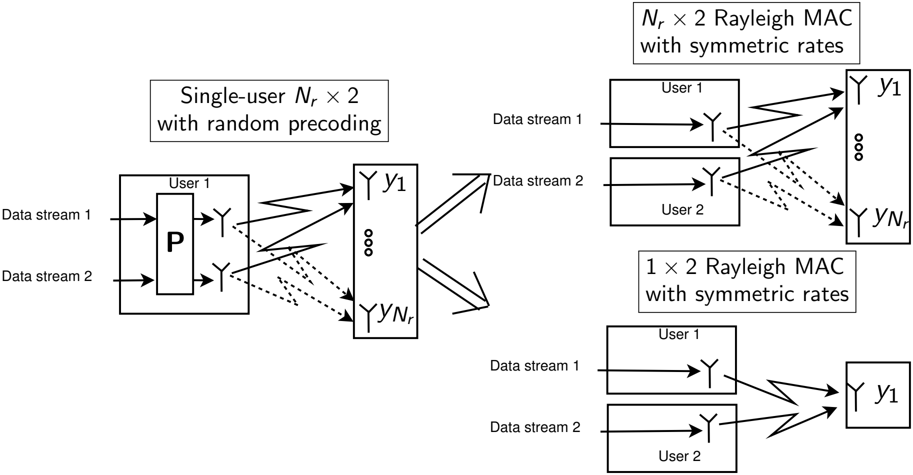
- We get

$$P^{\mathrm{WC}}_{\mathrm{out,C_{sym}}}(C,\Delta C) = 1-\sqrt{1-2^{-\Delta C}}.$$

Performance Extensions

- Explicit expression can be calculated for $N_r \times 2$ when **random** space-time precoding is applied
- This extension relies on the fact that the singular values of a sub-matrix of a unitary matrix have Jacobi distribution

Setup Extensions



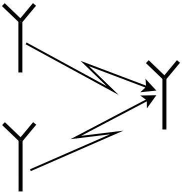
We stick to $N_t = 2$

What is so special about i.i.d. Rayleigh fading?

- Yes, it's widely used, but this is not the point...
- What is crucial for our purposes is that the precoding matrix $\mathbf{P}$ is built into the ensemble:
  - $\mathbf{H} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^H$
  - Edelman & Rao '04 - both $\mathbf{U}$ and $\mathbf{V}$ belong to the CUE (Haar measure)
- Symmetric rate Rayleigh MAC $N_r \times 2$:
  - What changes?
  - Now we don't minimize over worst case (ρ_1, ρ_2) pair, rather needs to take the expectation
- Symmetric rate Rayleigh MAC 1×2:
  - All that is left from the Rayleigh statistics (given the capacity) is the random matrix $\mathbf{P}$

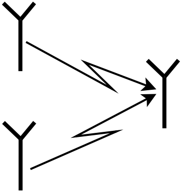
Rayleigh MAC with Symmetric Rates

- MAC: $$y = \sum_{i=1}^{N_t} h_i x_i + z$$



Rayleigh MAC with Symmetric Rates

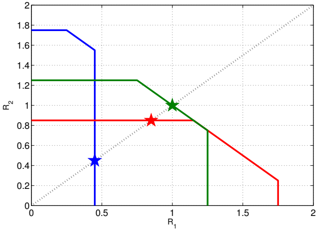
- MAC: $$y = \sum_{i=1}^{N_t} h_i x_i + z$$



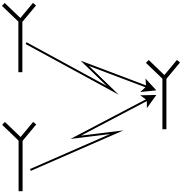
- CSI at Rx
- Equal average transmission power per antenna: $P = 1$
- $z \sim \mathcal{CN}(0,1)$
- $h_i \sim \sqrt{\mathrm{SNR}} \cdot \mathcal{CN}(0,1)$ and i.i.d.

Rayleigh MAC with Symmetric Rates

- Capacity region: $\sum_{i \in S} R_i \leq \log\left(1 + \sum_{i \in S} |h_i|^2\right)$ for all $S \subseteq \{1, 2, \ldots, N_t\}$
- Symmetric-rate capacity: $C_{\text{sym}} = \min_{S \subseteq \{1,2,\ldots,N_t\}} \frac{N_t}{|S|} \log\left(1 + \sum_{i \in S} |h_i|^2\right)$
- Sum-capacity $C_{\text{sum}} = \log\left(1 + \sum_{i=1}^{N_t} |h_i|^2\right)$



Simple MAC Transmission Protocol



Theorem 3

For a 1×2 Rayleigh MAC with sum capacity C_{sum}:

$$\Pr(C_{\mathrm{sym}} < R | C_{\mathrm{sum}}) = 2 \cdot \frac{2^{R/2} - 1}{2^{C_{\mathrm{sum}}} - 1}; \; 0 \leq R \leq \boldsymbol{C_{\mathrm{sum}}}$$

The chance of achieving C_{sum} can be calculated. For example:

$$\Pr(C_{\mathrm{sym}} = C_{\mathrm{sum}} | C_{\mathrm{sum}} = 2) = 1 - 2 \cdot \frac{2^{2/2} - 1}{2^2 - 1} = \frac{1}{3}$$

Simple MAC Transmission Protocol

Proof Follows Derivation of Theorem 2

- Note that in the SVD we have single singular value
- Recall

$$(*)\ \Pr\left(C(\{1\}) < R\right) = \frac{2^{R/2} - 1 - \rho_2}{\frac{2^C}{1+\rho_2} - 1 - \rho_2}$$

- Substitute $\rho_2 = 0$ in $(*)$ gives the theorem

1×2 Rayleigh MAC with Symmetric Rates

What about a practical scheme?

- We would like a scheme for which the outage behaves as Theorem 3:

$$-\log\left(\Pr(C_{\text{sym}} < R | C_{\text{sum}})\right) \approx -\log\left(2 \cdot \frac{2^{R/2}-1}{2^{C_{\text{sum}}}-1}\right)$$
$$\approx -\log\left(2 \cdot 2^{-(C_{\text{sum}}-R/2)}\right) \quad (\text{for } 2^{R/2} \gg 1)$$
$$\approx (C_{\text{sum}} - R/2)$$

- Recall that for $N_r \times 2$ ($N_r \geq 2$), we had:

$$-\log\left(\Pr(C_{\text{sym}} < R | C_{\text{sum}})\right) \approx (C_{\text{sum}} - R)$$

$\Longrightarrow$ for 1×2 ML behavior is now changed

Does integer-forcing still get the job done?

Does the achievable rate (of IF) have (qualitatively) the same (improved) behaviour?

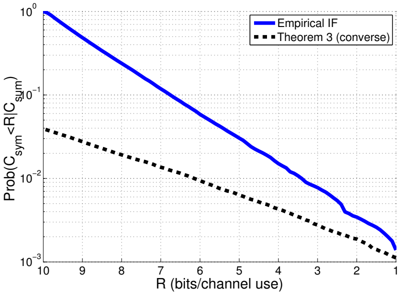


Figure: ML vs. IF for 1×2 channel with $C_{\text{sum}} = 10$

1×2 Rayleigh MAC with Symmetric Rates

Is there a problem with IF?

Remember the MAC-DMT moral...

Summary and Outlook

Summary

- Explicit lower bounds on integer-forcing outage probability
  - Compound (worst-case) single user $N_r \times 2$ ($N_r \geq 2$)
  - Rayleigh MAC $N_r \times 2$ ($N_r \geq 2$) - path to analysis
  - Rayleigh MAC 1×2

Outlook

- Rayleigh MAC $N_r \times 2$ ($N_r \geq 2$) - derive explicit expressions
- Rayleigh MAC 1×2
  - Can IF performance be improved (to match ML behaviour)?
  - We believe it can (lessons from MAC-DMT...)
- What about $N_t > 2$?
  - Same principles are applicable?
- Stay tuned...

Thanks for your attention!